ODE Refresher

1st-order linear

$$y'(t) + p(t)y = g(t)$$

solved by using an integrating factor

$$\mu(t) = e^{\int p(t)dt}$$ leading coefficient is 1

multiply both sides by that

left side is the derivative of $\mu(t)$ and $y(t)$

$$\underbrace{\mu y' + \mu p y} = \mu g$$

$$\frac{d}{dt}(\mu y) = \mu g$$

integrate and solve for y

example

$$y' + \frac{2}{t} y = \frac{\cos(t)}{t^2} \qquad (t>0)$$

$$\mu = e^{\int \frac{2}{t} dt} = \cancel{e^{2 \ln t}} = e^{2 \ln t} = e^{\ln t^2} = t^2$$

multiply both sides by that

$$t^2 y' + 2t y = \cos(t)$$

$$\frac{d}{dt}(t^2 y) = \cos(t)$$

integrate

$$t^2 y = \sin(t) + C$$

$$\boxed{y = \frac{\sin(t)}{t^2} + \frac{C}{t^2}}$$

1st-order linear will make an appearance later when we solve ~~the~~ heat equation : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ $\quad u(x,t)$

1st-order shows up in the context of the heat eq. as

$$T' + \alpha^2 \lambda T = 0 \qquad T = T(t) \qquad \alpha, \lambda \text{ constants, positive}$$

this can be solved by using an integrating factor or as a separable eq.

$$\frac{dT}{dt} + \alpha^2 \lambda T = 0$$

$$\frac{dT}{dt} = -\alpha^2 \lambda T$$

$$\frac{dT}{T} = -\alpha^2 \lambda \, dt \qquad \text{integrate both sides}$$

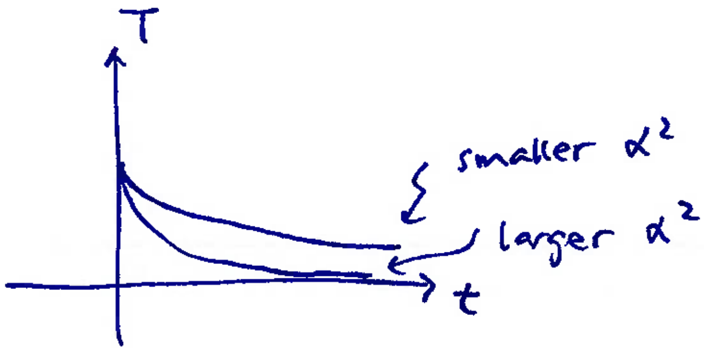
$$\ln|T| = -\alpha^2 \lambda t + C$$

$$T = e^{-\alpha^2 \lambda t + C} = e^{-\alpha^2 \lambda t} \cdot e^{\overbrace{C}^{\text{constant}}}$$

$$\boxed{T(t) = Ce^{-\alpha^2 \lambda t}}$$

this governs the time part of the solution of heat eq.

for a given λ, the larger α^2 is, the faster T decays



2nd-order homogeneous

$$y'' + Ky = 0 \qquad K : \text{constant}$$

solutions are of the form e^{rt}

where r is the solution to the characteristic eq.

$$\boxed{r^2 + K = 0}$$

solution depends on what K is

if $K<0$, $r = \pm\sqrt{K}$

solutions are $y = e^{\pm\sqrt{K}t}$

grows w/o bound

general solution: $y = c_1 e^{\sqrt{K}t} + c_2 e^{-\sqrt{K}t}$

or $y = d_1 \cosh(\sqrt{K}t) + d_2 \sinh(\sqrt{K}t)$

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ $\quad \sinh(x) = \dfrac{e^x - e^{-x}}{2}$

if $K=0$ $\quad y'' = 0$

so, $y = c_1 + c_2 t$ (just a line)

if $K>0$ $\quad r = \pm i\sqrt{K}$

solutions are $y = \cos(\sqrt{K}t)$ and $y = \sin(\sqrt{K}t)$

general solution: $y = c_1 \cos(\sqrt{K}t) + c_2 \sin(\sqrt{K}t)$

<u>bounded</u> solution

we will see that when solving the heat, wave, and Laplace's eqs.

it shows up as

$$X''(x) + \lambda X(x) = 0 \quad \text{or} \quad Y''(y) + \lambda Y(y) = 0$$

uppercase X

solutions:

$$X(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x) \quad \text{if } \lambda > 0$$

$$X(x) = C_1 \cosh(\sqrt{\lambda}\, x) + C_2 \sinh(\sqrt{\lambda}\, x) \quad \text{if } \lambda < 0$$

$$X(x) = C_1 + C_2 X \quad \text{if } \lambda = 0$$

in the context of the heat eq, this is the <u>space</u> part of the solution (it tells us how temperature varies w/ position with time fixed)

next, let's review nonhomogeneous 2nd-order eqs.

in the context of mass-spring system : $m x'' + k x = f(t)$

we will focus on sinusoidal $f(t)$: $f(t) = F_0 \sin(\omega t)$

constant

frequency of input

general solution: $x(t) = x_c + x_p$

complementary (right side is 0)

particular (due to the right side)

for $m x'' + k x = 0$

$$x_c = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$\sqrt{\frac{k}{m}} = \omega_0$ natural frequency.

if $f(t) = F_0 \sin(\omega t)$ and $\omega \neq \omega_0$

by the method of undetermined coefficients

assume $x_p = A \cos(\omega t) + B \sin(\omega t)$ always both cosine and sine

Sub into $mx'' + kx = F_0 \sin(\omega_0 t)$

$\vdots$

$A = 0, \quad B = \dfrac{F_0}{k - m\omega^2}$ trouble if $k = m\omega^2$ or $\omega = \sqrt{\dfrac{k}{m}}$

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{F_0}{k - m\omega^2} \sin(\omega t)$$

if $\sqrt{\dfrac{k}{m}} = \omega = \omega_0$ (natural = input $\longrightarrow$ resonance)

adjustment to particular: $x_p = A\,\underline{\underline{t}} \cos(\omega t) + B\,\underline{\underline{t}} \sin(\omega t)$

Sub into $mx'' + kx = F_0 \sin(\omega_0 t)$

$\vdots$

$$x_p = -\frac{F_0}{2m\omega}\,\underline{\underline{t}} \cos(\omega t)$$

blows up as $t \to \infty$